Modeling Blood-Vessel Walls and its Interaction with Blood Flow to the Brain

Dr. Jacques C. Richard, JC-Richard@CSU.edu

Dr. Samuel P. Bowen, S-Bowen@Csu.edu

Dr. Paul F. Fischer, fischer@mcs.anl.gov

Lateef D. Muhammad, Bran569341@aol.com

Two Ideas

1. Deformation of upper layer of orbit of eye
2. Deformation of artery into brain while simultaneously measuring velocity, radius, blood pressure, etc.

Problem

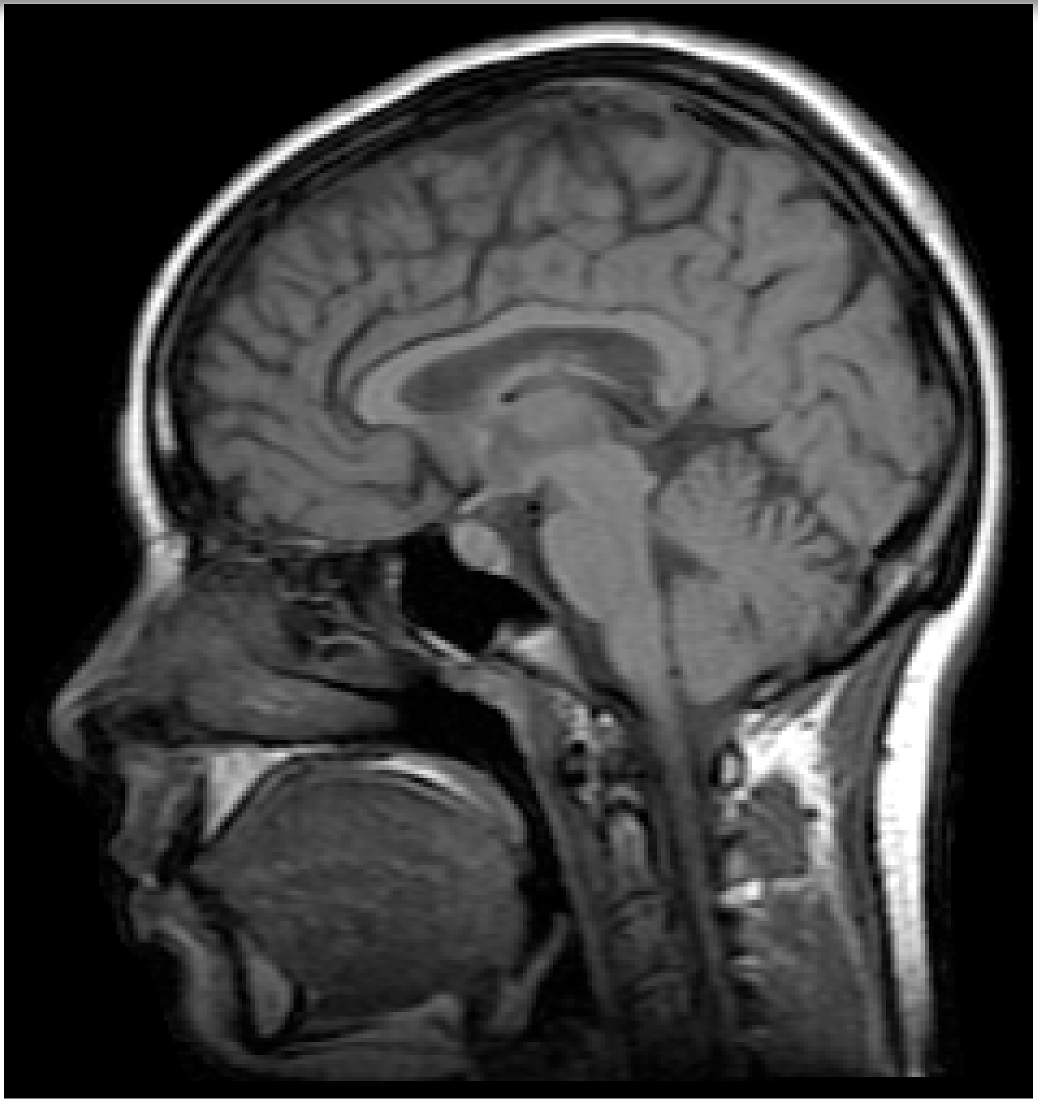
- Modeling blood flow to brain requires dealing with complicated interaction of blood with flexible blood vessel wall
- Simulating transitional flows, wherein transition from one flow type to another, as in laminar to turbulent, is complicated enough
- But in complex geometries such as human arteries, it is further complicated.
- Adding further that such complex geometries can move or deform is even more challenging.

Specific Goal

- Wall vibration is a possible energy dissipation mechanism in transitional flows adjacent to compliant surfaces
- Investigate such combined fluid-structure interactions with a spectral element code

Brain MRI

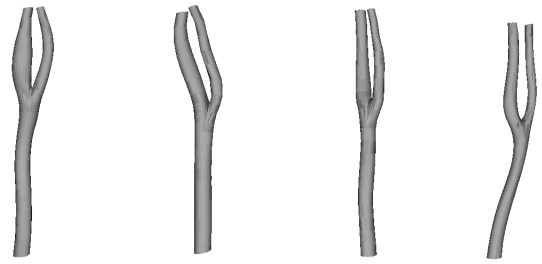
Blood flow thru brain: too complex



Analyze flow in/out thru carotid artery

Current Models

- Fischer *et* al. model transition in a stenosed carotid artery: http://www-unix.mcs.anl.gov/~fischer/car_trans.html
- Taylor; *et al*. design surgical support system with FEM and MRI
- Xu used carotid artery bifurcation models reconstructed from MR scans of 3 healthy adults.



Current Models

- Blood is a complex fluid in that different types of blood cells make its modeling difficult.
- However, it could be modeled as an incompressible Newtonian fluid for the limited distance from the heart to the brain

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet \nabla \mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}$$

$$\nabla \bullet \mathbf{u} = 0$$

Analysis

- Expand Fischer *et al.*'s work by using a compliant wall
- First step: validate compliant wall modeling with simpler 2D geometry
  - Model free surface with spectral element code
  - Model moving wall
  - Model combinations, etc.

Spectral Element Method

- Like Finite Element Method
- But with Spectral Functions
- Infinitely differentiable global functions of SEM vs. local character of FEM functions
- Adaptive mesh
- Polynomials of high and differing degrees
- Non-conforming spectral element method as described by Fischer; Patera; van de Vosse and Minev; Bernadi and Maday; with Ho's free-surface & moving wall extensions.

SEM Discretization

- Polynomial approximation for velocity two degrees higher than that for pressure
- Avoids spurious pressure modes.
- Like solving eqs. on a staggered grid where **u** and *p* are solved on different grids but coupled (e.g., via interpolation)

SEM Approach

- Temporal discretization of Navier-Stokes eqs. based on high-order operator splitting methods
  - Splitting problem into convection & diffusion
  - Some combination of integration schemes for convection operator or for time-dependent terms that may be high order
  - With some degree of polynomial for SEM discretization of diffusion terms giving high-order in space
- Coupled w/SEM spatial discretization to yield sequence of symmetric positive definite (SPD) sub-problems to be solved at each time step.

SEM Algorithm

- Stokes discretization based on variational form: Find $(\mathbf{u}, p)$ in $X \times Y$ such that

$$\frac{1}{\mathrm{Re}}(\nabla \mathbf{u}, \nabla \mathbf{v}) + \frac{3}{2\Delta t}(\mathbf{u}, \mathbf{v}) - (p, \nabla \bullet \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

$$(\nabla \bullet \mathbf{u}, q) = 0$$

- $\forall\ (\mathbf{v}, q) \in X \times Y$, I.e., as weights in $X \times Y$.
- Inner products: $(l,g)=\int_{\Omega(t)} l(\mathbf{x})\, g(\mathbf{x})\, d\,\mathbf{x}$

Discrete Stokes System

- Inserting SEM basis

  $f(\mathbf{x}^k(\mathbf{r}))|_{\Omega^k} = \sum^{N}_{i=0} \sum^{N}_{j=0} f^k_{ij}\, h_i(r_1)\, h_j(r_2)$

  into

  $$\frac{1}{\mathrm{Re}}(\nabla \mathbf{u}, \nabla \mathbf{v}) + \frac{3}{2\Delta t}(\mathbf{u}, \mathbf{v}) - (p, \nabla \bullet \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

  $$(\nabla \bullet \mathbf{u}, q) = 0$$

  yields $H\ \underline{\mathbf{u}}^n - D^T \underline{p}^n = B\ \underline{\mathbf{f}}^n$, $D\ \underline{\mathbf{u}}^n = 0$

  where

  - ✓ $H = A/\mathrm{Re} + B/\Delta t$ = discrete equivalent of Helmholtz operator;
  - ✓ A = discrete Laplacian,
  - ✓ B = mass matrix associated with the velocity mesh (diagonal);
  - ✓ D = discrete divergence operator

Proper Subspaces

- The proper subspaces for **u**, **v**, and p, q are:

  $X=\{\mathbf{v} : v_i$ □ $H^1{}_0$(□), $i=1,\ldots,d$, $\mathbf{v} = 0$ on $\partial\Omega_v\}$, d=2 if 2D...

  $Y= L^2$(□)

  - L^2 is the space of square integrable functions on □;
    $\int_\Omega v^2 \mathrm{dV} = \int_\Omega v^2 \mathrm{d}^3\mathbf{r}$
  - $H^1{}_0$ is the space of functions in L^2 that vanish on the boundary ($_0$) and whose first derivative (^{1}) is also in L^2;
    $\int_\Omega (\partial v/\partial \mathrm{r})^2 d\mathrm{V} = \int_\Omega (\partial v/\partial \mathrm{r})^2 d^3\mathbf{r}$

- Spatial discretization proceeds by restricting **u**, **v**, and p, q to compatible finite-dimensional velocity and pressure subspaces: X^N □ X and Y^N □ Y

Handling Pressure

- To avoid spurious pressure modes, Maday, Patera and Rønquist, and, Bernardi and Maday suggest different approximation spaces for velocity and pressure:

  $X^N = X \cap \mathbf{P}_{N,K}(\Omega)$

  $Y^N = Y \cap \mathbf{P}_{N-2,K}(\Omega)$

  where

$$\mathbf{P}_{N,K}(\Omega) = \{ v(\mathbf{x}^k(\mathbf{r}))|_{\Omega^k} \in \mathbf{P}_N(r_1) \otimes \ldots \otimes \mathbf{P}_N(r_d),\ k=1,\ldots,K \}$$

  and $\mathbf{P}_N(r)$ is space of all polynomials of degree $\leq N$

Space Dimensions

- Dimension of Y^N is $K(N\text{-}1)^d$ since continuity is enforced for functions in Y^N
- Dimension of X^N is $dK(N+1)^d$ because
  - functions in X^N must be continuous across sub-domain interfaces
  - Dirichlet bc's on $\partial\Omega_v$

Function Spaces

- <u>Velocity Space</u>: Basis chosen for $\mathbf{P}_N(r)$ is set of Lagrangian interpolants on Gauss-Lobatto-Legendre (GL) quadrature pts. in ref. domain: $\square_i$ $\square$ $[\square 1,1]$, $i=0,\ldots,N$
- <u>Pressure Space</u>: Basis chosen for $\mathbf{P}_{N\text{-}2}(r)$ is set of Lagrangian interpolants on Gauss-Legendre (G) quadrature pts. in ref. domain: $\square_i$ $\square$ $]\square 1,1[$, $i=1,\ldots,N\text{-}1$
- Basis for velocity is continuous across sub-domain interfaces but basis for pressure is not

SEM Algorithm Quadrature

- Subscripts $(.,.)_{GL}$ and $(.,.)_G$ referred to Gauss-Lobatto-Legendre (GL) and Gauss-Legendre (G) quadrature which are:
- $\int^1_{-1} f(x)dx = w_1 f(-1) + w_N f(1) + \sum^N_i w_i f(x_i)$

Gauss-Lobatto-Legendre (*GL*) Quadrature

- $\int^1_{-1} f(x)dx = w_1 f(-1) + w_N f(1) + \sum^n_i w_i f(x_i)$ where

$$w_i^{GL} = \frac{2N}{(1 - x_i^2)L''_{N-1}(x_i)L'_M(x_i)} = \frac{2}{N(N-1)[L_{N-1}(x_i)]^2}$$

- L_n are the *Legendre* polynomials, Gauss-Lobatto points are zeroes of L'_N or $(1-x^2)\, L'_N$ & at endpoints (-1,1)

$$w_{1,N}^{GL} = \frac{2}{N(N-1)}$$

Gauss-Lobatto-Legendre (*GL*) Quadrature

- w/error

$$E = \frac{N(N-1)^3 2^{2N-1}[(N-2)!]^4}{(2N-1)[(2N-2)!]^3} f^{(2N-2)}(\xi)$$

- for $\xi \in$ (-1,1)
- The weights may also be written as

$$w_i^{GL} = \rho_i = \frac{2}{N(N+1)} \frac{1}{[L_N(x_i)]^2}$$

Gauss-Legendre (*G*) Quadrature

- Same as Gauss-Legendre-Lobatto <u>but</u> w/o endpoints (<u>not</u> used for prescribed function values at boundaries) and the weights are

$$w_i^G = \square_i = \frac{2}{(1 - x_i^2)[L_{N+1}(x_i)]^2}$$

- Where L_N are the *Legendre* polynomials, the Gauss points (interior points) are zeroes of L_{N+1}

Interpolation Polynomials

- Basis functions are Legendre-Gauss-Lobatto-Lagrange interpolation polynomials:

$$h_i = \frac{-1}{N(N+1)L_N(x_i)} \frac{(1-x^2)L_N^{'}(x)}{x-x_i}$$

Free Surface Model

- At moving boundary ($\partial\Omega_{□}(t)$, also traction boundary) btwn fluid & other continuum is Neumann bc, $□_{ij}\, n_j = T_i = □ \cdot \mathbf{\hat{u}}_n = \mathbf{T}$, the traction.
- Surface tension (□) modeled at this interface by adding $\int_{\partial\Omega□(t)} □\, □\, v_i\, n_i\, dA$ to RHS of Stokes problem
  - □ $\partial\Omega_{□}(t)$ is for traction boundary,
  - ♣ □ = curvature

Free Surface Model

- In 2D, $\square = s_{i,\square}$,
  - s_i is a unit vector tangent to free surface segment from pt. a to b,
  - $\square$ is a curvilinear coord. along segment
- So that $\int_{\partial\Omega\square(t)} \square\, \square v_i\, n_i\, dA = \int_a^b \square\, v_i\, s_{i,\square} d\square$
- Integrating by parts, noting that $v_{i,\square} = v_{i,j}\, s_j$,

$$\int_{\partial\Omega\square(t)} \square\, \square v_i\, n_i\, dA = - \int_a^b \square\, v_{i,j}\, s_i\, s_j\, d\square + (\square\, v_i\, s_i)_a^b$$

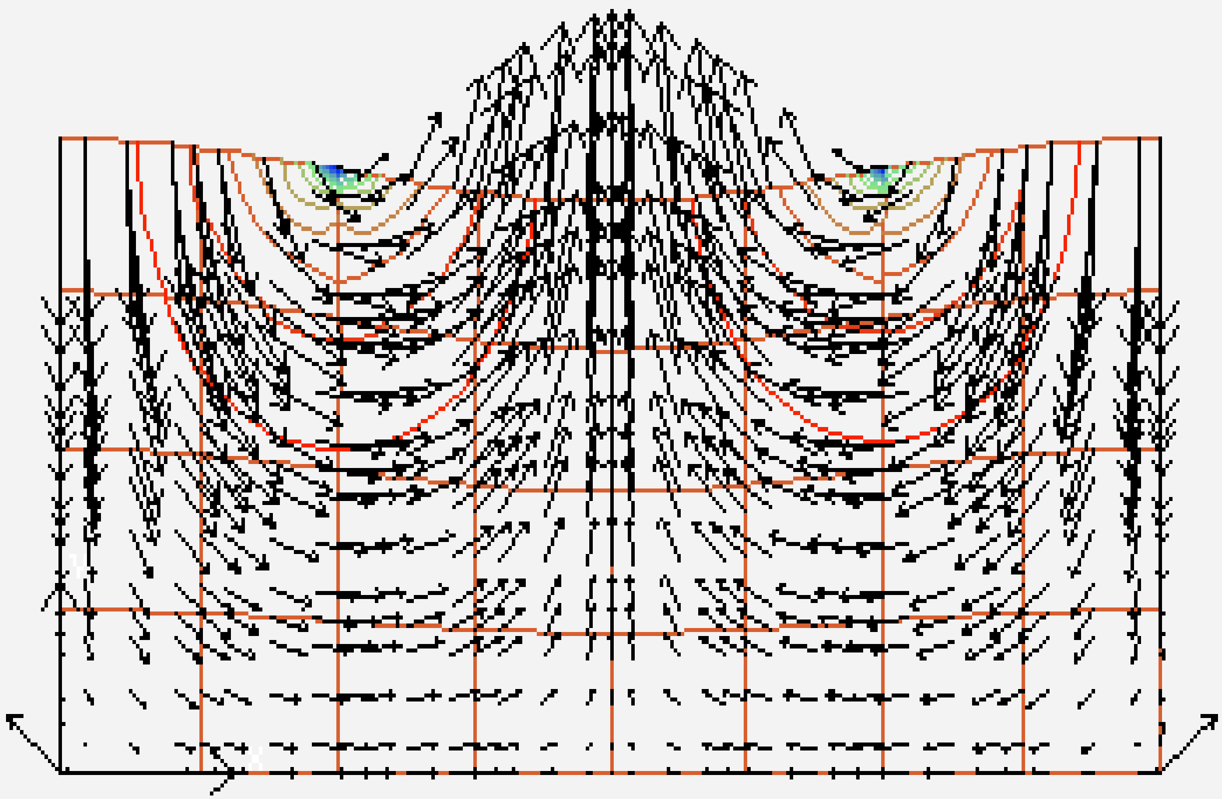
Moving Mesh

- Account for moving mesh by adding another term to RHS of Stokes' problem:
  $\int_{\Omega(t)} v_i \, (u_i \, w_j \,),_j \, dV = \int_{\Omega(t)} \mathbf{v} \, \square \cdot (\, \mathbf{u} \, \mathbf{w} \,) \, dV =$
  $(\mathbf{v} \, , \square \cdot (\, \mathbf{u} \, \mathbf{w} \,) \,)$ where **w** is the mesh velocity
- The mesh velocity provides for nodal coords. to be updated as $d\mathbf{x}_{\mathrm{node}}/\mathrm{dt} = \mathbf{w}$

Results

- Free surface, initially sinusoidal in shape
- Then allowed to vary according to differences in initial fluid pressure compared to that above the free surface
- Free surface gradually began to move and be reshaped by the pressure imbalance

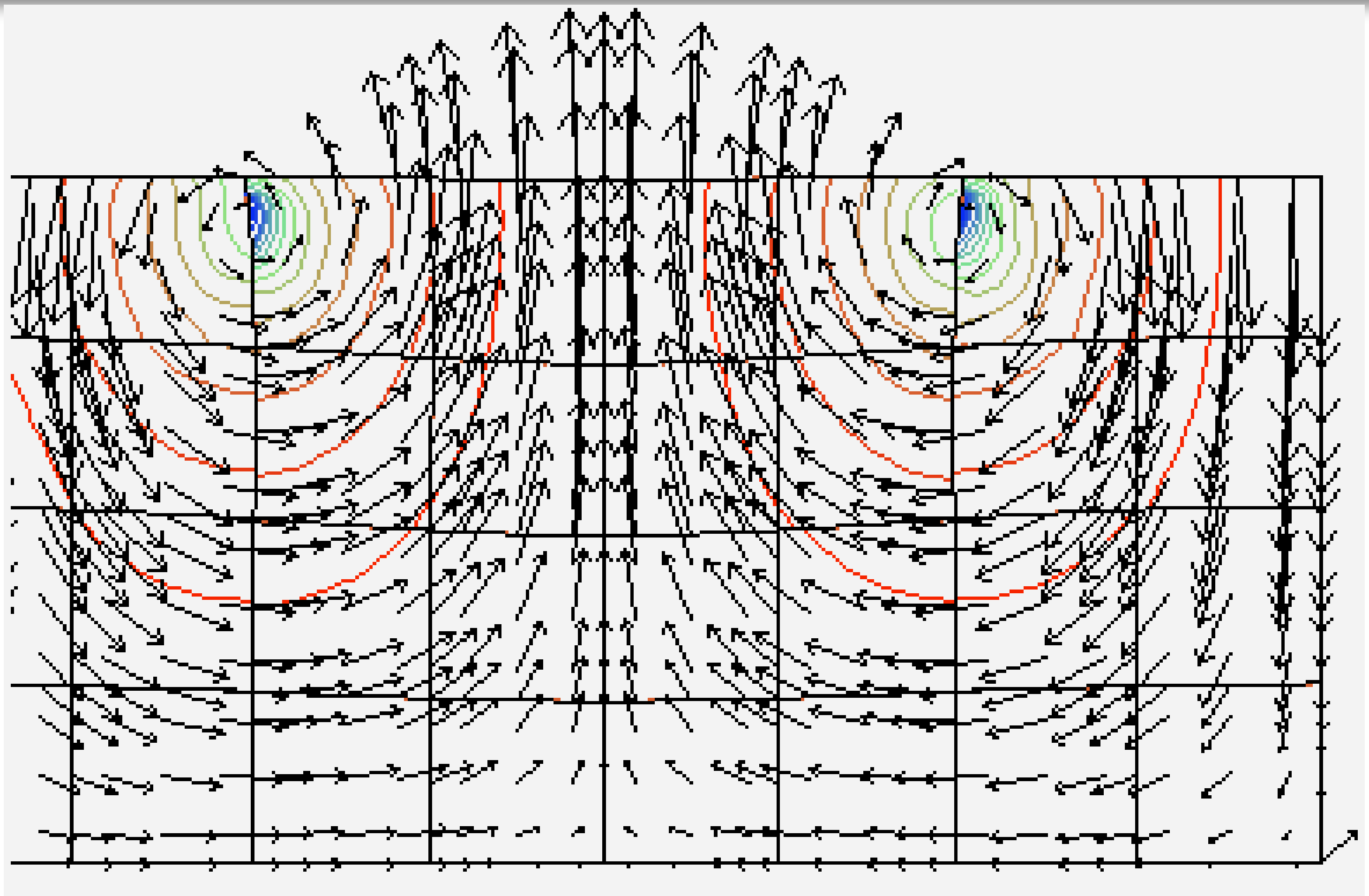
Free Surface



At dimensionless t=0.8, 200th time step; fluid pushing up the middle…

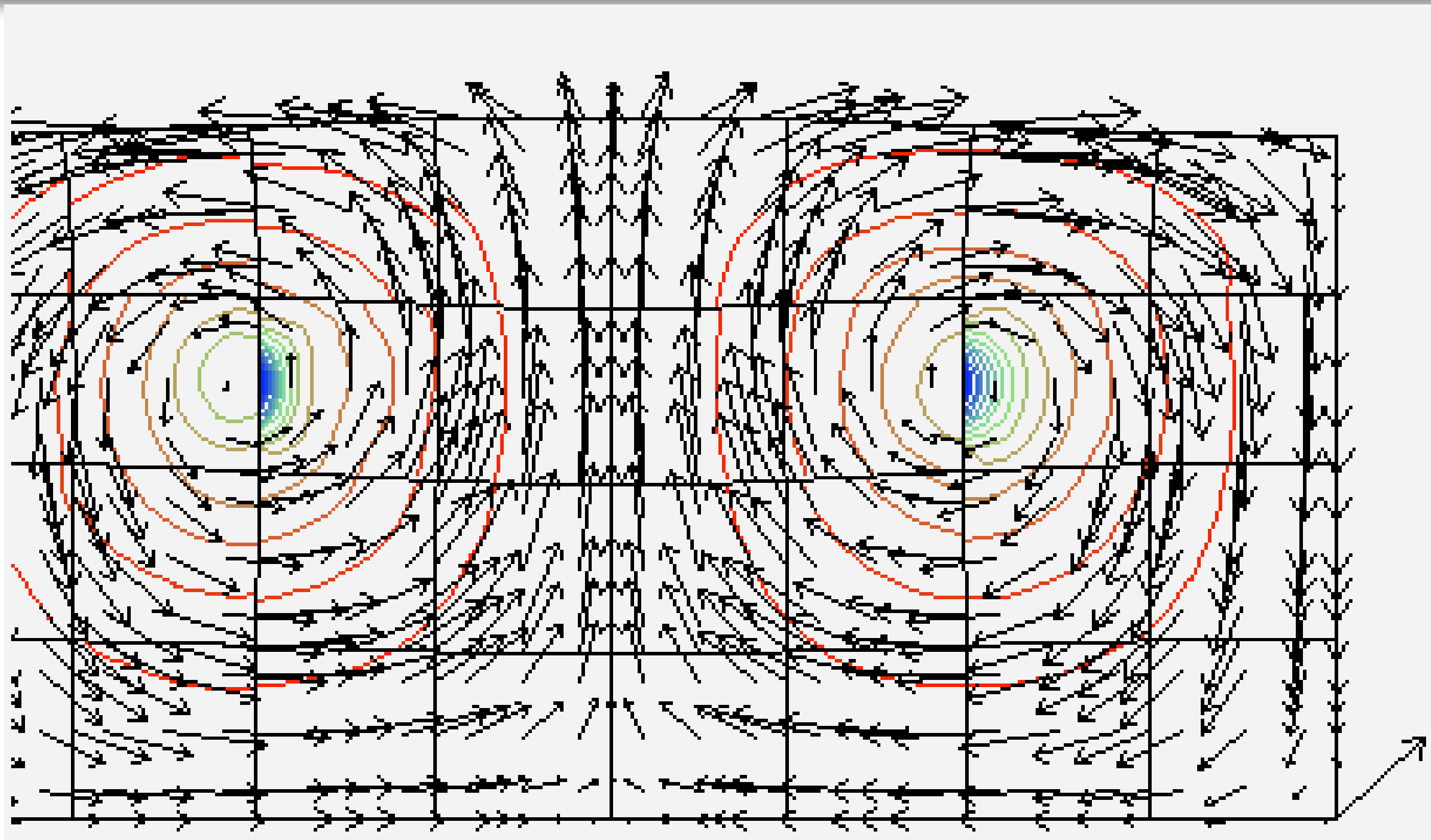
K=32 elements, N=5th order

Free Surface



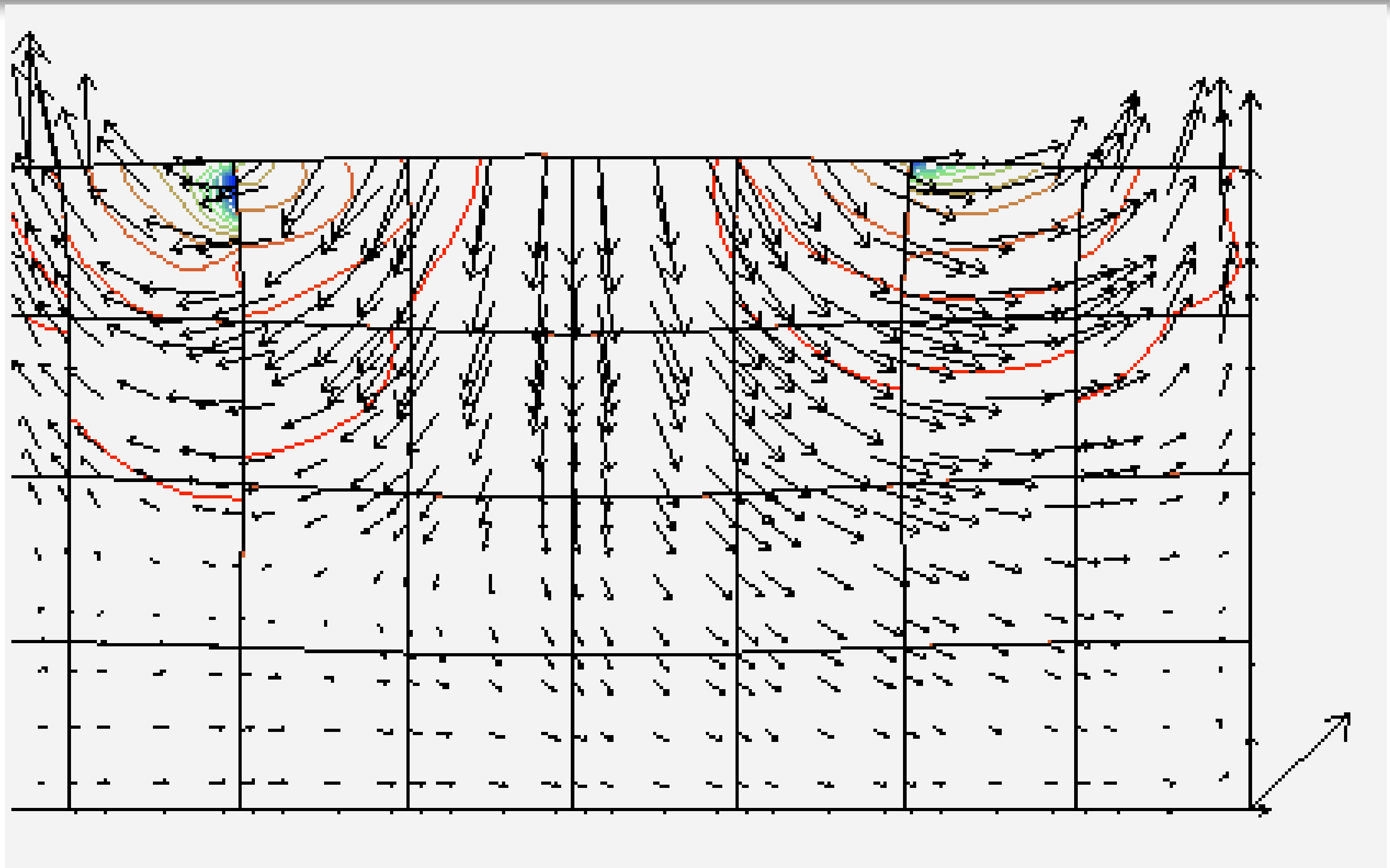
At dimensionless t=1.6, 400th time step; appears to be flattened but fluid still pushing up, vortex pair forming near free surface…

Free Surface



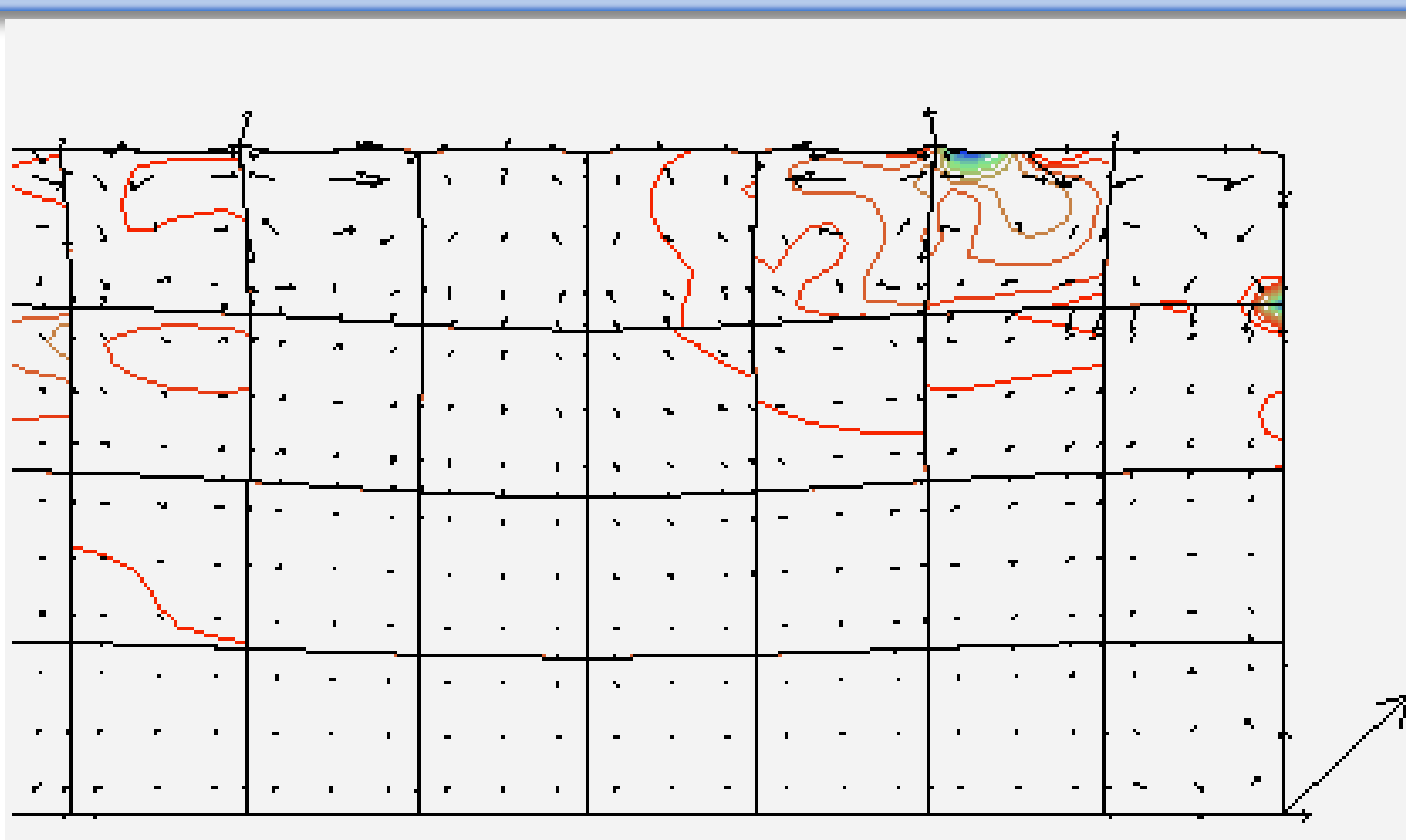
At dimensionless t=2.4, 600th time step; appears to be flattened but fluid pushed to sides at top, twin vortices moved down…

Free Surface



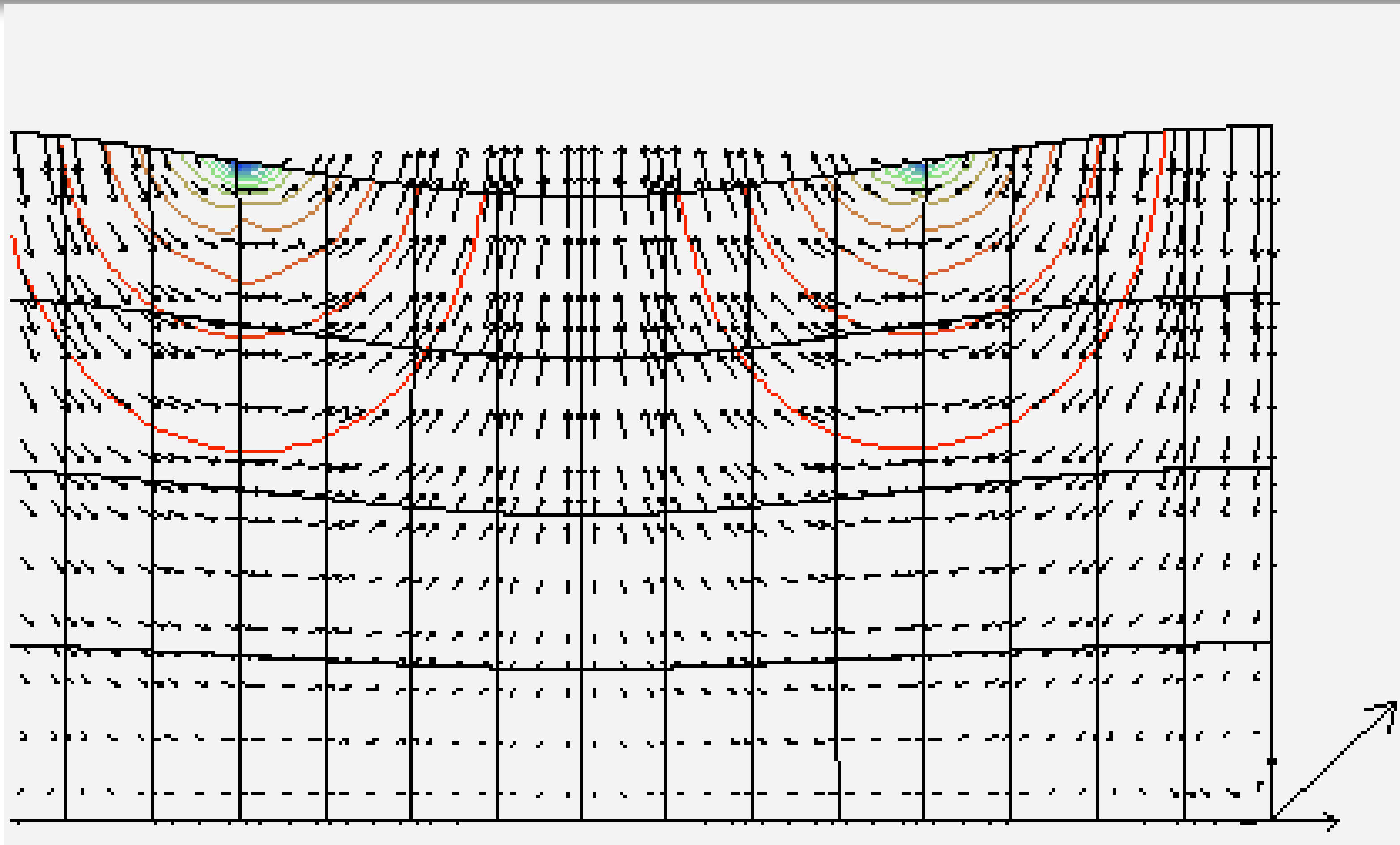
At dimensionless t=3.2, 800^{th} time step; not flattening but seeming to be reversing shape like an oscillating free surface; vortex pair back up…

Free Surface



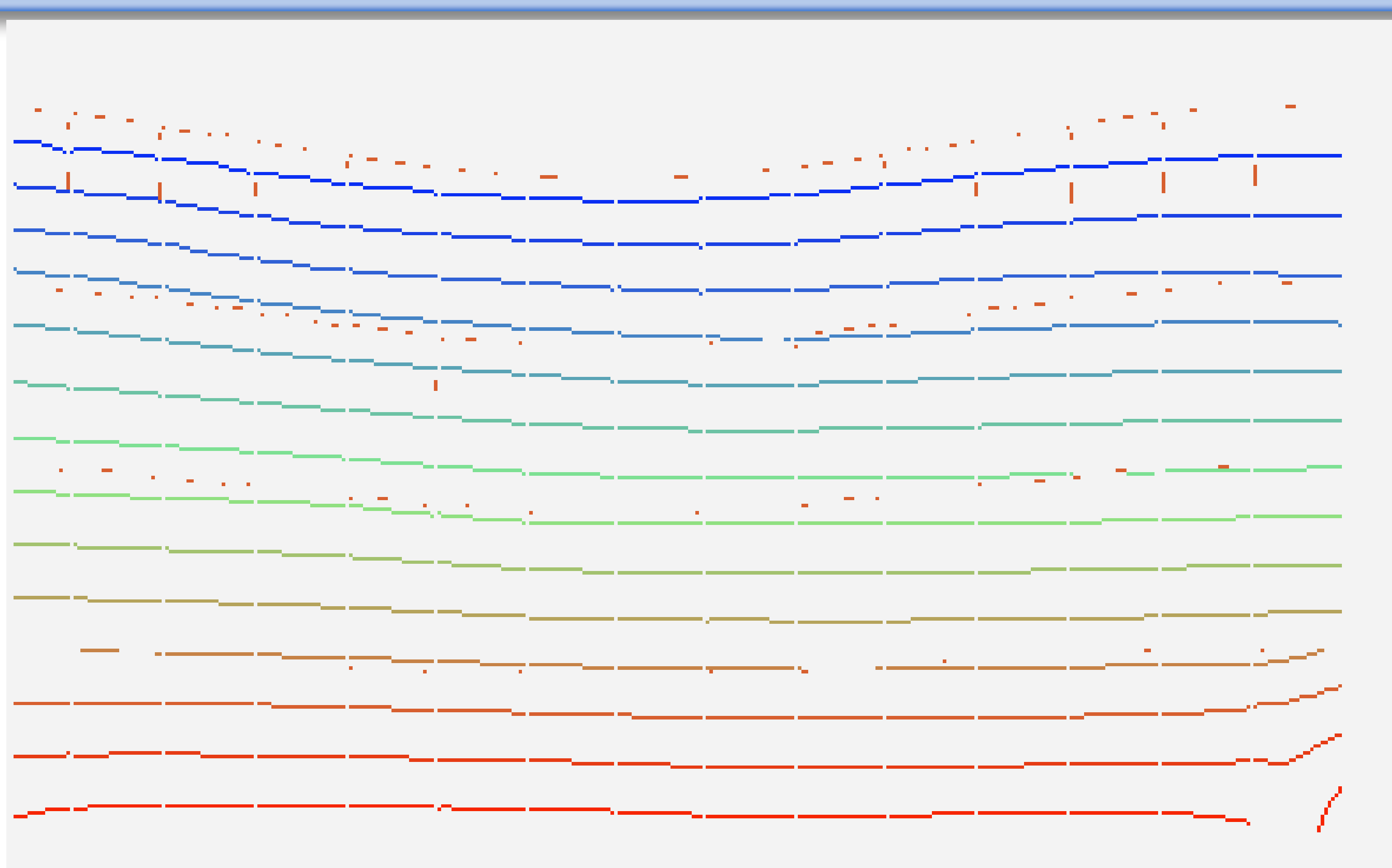
At dimensionless t=6.4, 1600th time step; not oscillating shape but free surface breaking up; vortices broken up then numerical instability

Free Surface: Finer Mesh



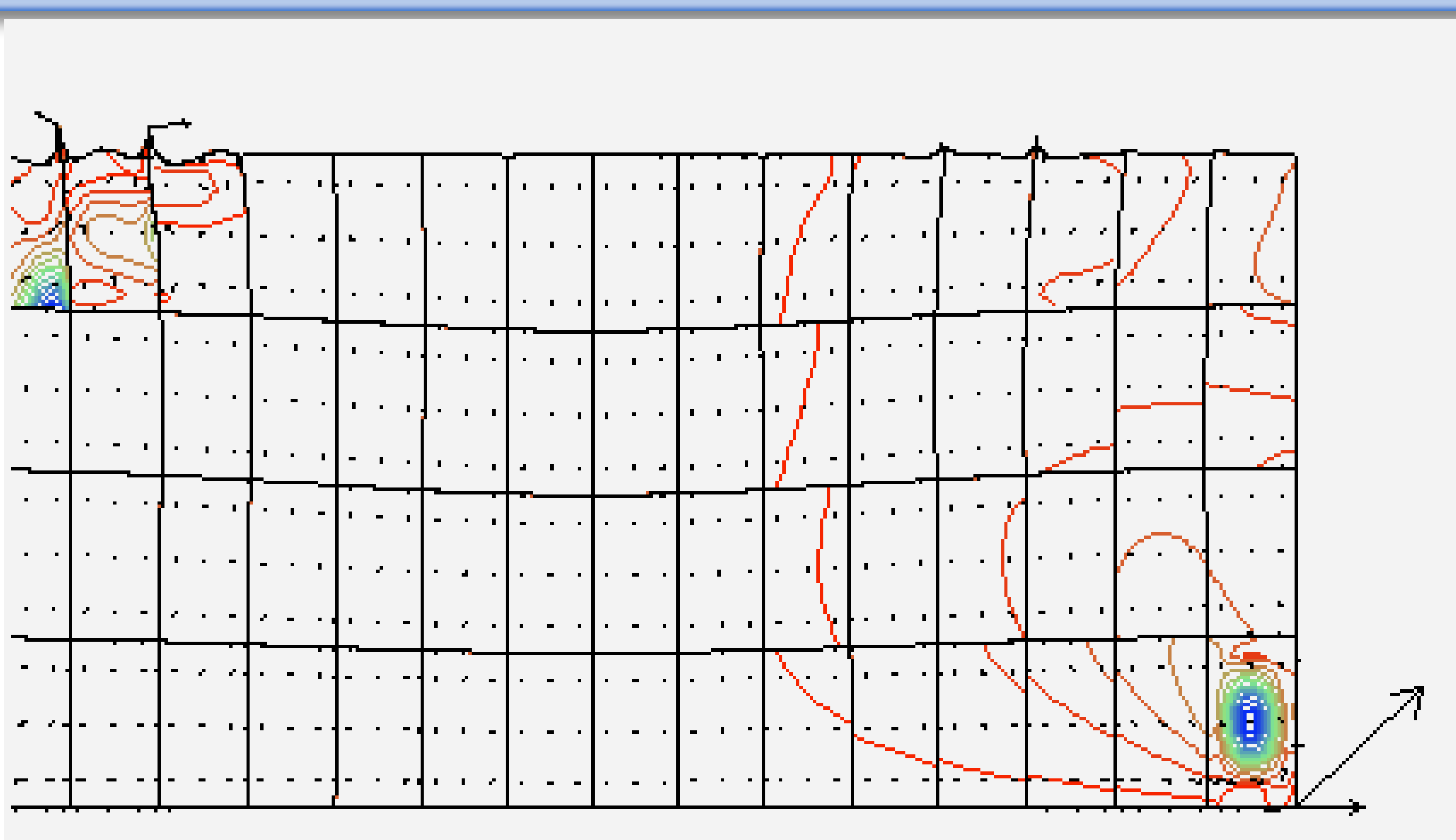
At dimensionless t=0.8, 200th time step; Finer mesh shows the same thing but with better resolution: K=64, N=5

Free Surface: Finer Mesh



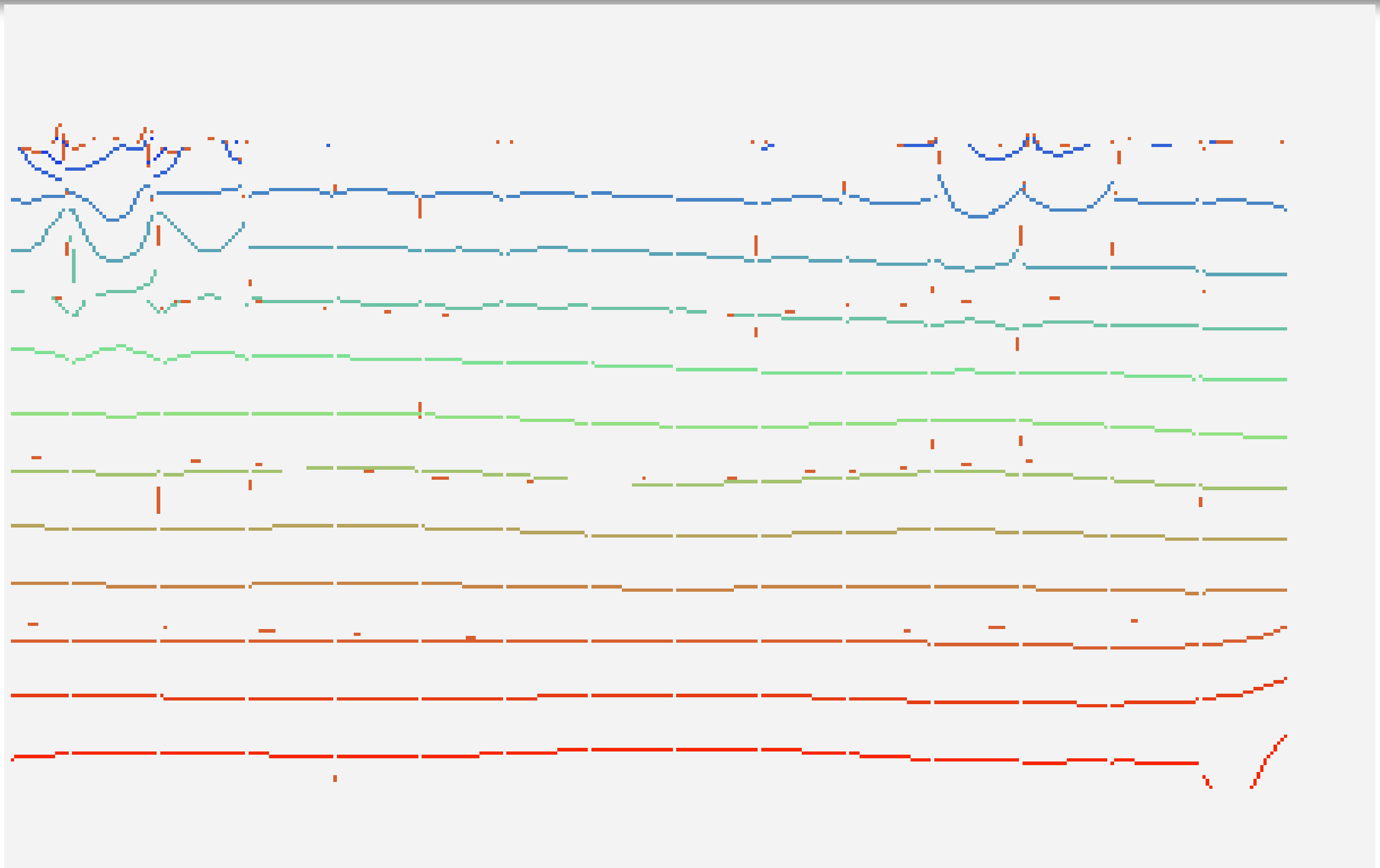
At dimensionless t=0.8, 200th time step; But isobars show pressure gradient oscillating w/free surface

Free Surface: Finer Mesh



At dimensionless *t*=24, 6000th time step; Finer mesh better details break-up

Free Surface: Finer Mesh



At dimensionless *t*=24, 6000th time step; Finer mesh better details break-up & accompanying pressure distortions

Shape-Changing Wall

- Free surface was replaced with a compliant wall that starts with a sinusoidal shape but that shape varied sinusoidally with time as well.
- The fluid response contained regions where the flow-field showed fluid circling up into the high points of the sinusoidal-shaped wall and down from the lower points.
- There appeared vortices oscillating with the oscillating wall

Moving wall

- Non-time-varying sinusoidal-shaped wall given a prescribed vertical velocity downward into the fluid region,
- The fluid moved out through an unconstrained end.
- However, fluid became entrapped under a hump of the sinusoidal shape.
- The simulation could not continue further beyond the point where the non-compliant sinusoidal-shaped wall reached the bottom of the fluid region

Conclusion

- The results obtained suggests that a compliant wall can have its shape altered by the fluid region it bounds and the reverse
- The results also show how the wall can determine where fluid can flow and the way that flow occurs

Future Work

- More combinations planned
- Extend to 3D
- Apply to carotid artery
- Extend SEM potential w/parallel numerical libraries w/solvers, pre-conditioners, etc.

Thanks to

- Dr. Linda Phaire-Washington for opportunity to be Faculty and Student Team (FaST).
- US Department of Energy (DOE), Argonne National Laboratory, MCS Division.
- National Science Foundation and Dr. Hicks, via Louis Stokes Alliance for Minority Participation (LS-AMP or Chicago AMP, ChAMP) program through Chicago State University (CSU) under Dr. Leroy Jones, for their support.